ОБОЗРЕНИЕ
ПРИКЛАДНОЙ И ПРОМЫШЛЕННОЙ
Том 23 МАТЕМАТИКИ Выпуск 2
2016

A. V. K a l i n k i n, **A. E. K u l z h a n o v a** (Moscow, Bauman Moscow State Technical University). **The probability of the extinction of branching process with the scheme of interaction $2T \to 3T$; $T \to 0$.**

We consider a time-homogeneous Markov process $\xi(t)$, $t \in [0,\infty)$, on the set of states $N = \{0,1,2,\dots\}$ with transition probabilities $P_{ij}(t) = \mathbf{P}\{\xi(t) = j \,|\, \xi(0) = i\}$. Let us suppose that the transition probabilities have the following form as $t \to 0+$, $\lambda_1 > 0, \lambda_2 > 0$,

$$P_{i,i-1}(t) = \lambda_1 it + o(t), \quad P_{ii}(t) = 1 - (\lambda_2 i(i-1) + \lambda_1 i)t + o(t),$$
$$P_{i,i+1}(t) = \lambda_2 i(i-1)t + o(t), \quad P_{ij}(t) = o(t),\ j \neq i-1, i, i+1.$$

Let us introduce the generating functions of the transition probabilities $F_i(t;s) = \sum_{j=0}^{\infty} P_{ij}(t)s^j$, $|s| \leqslant 1$. The second (forward) system of Kolmogorov differential equations for the transition probabilities of the process $\xi(t)$ is equivalent to the partial differential equation [2], [4], [5],

$$\frac{\partial F_i(t;s)}{\partial t} = \lambda_2(s^3 - s^2)\frac{\partial^2 F_i(t;s)}{\partial s^2} + \lambda_1(1-s)\frac{\partial F_i(t;s)}{\partial s}, \quad F_i(0;s) = s^i.$$

We introduce an exponential generating function $G_j(t;z) = \sum_{i=0}^{\infty}(z^i/i!)P_{ij}(t)$. The first (backward) system of differential equations for the transition probabilities is equivalent to the partial differential equation [2],

$$\frac{\partial G_j(t;z)}{\partial t} = \Big[\lambda_2 z^2\Big(\frac{\partial^3}{\partial z^3} - \frac{\partial^2}{\partial z^2}\Big) + \lambda_1 z\Big(1 - \frac{\partial}{\partial z}\Big)\Big]G_j(t;z), \quad G_j(0;z) = \frac{z^i}{i!}.$$

The state 0 is absorbing. Introduce the probability of extinction $q_{i0} = \lim_{t\to\infty} P_{i0}(t)$, $i \in N$, and the generating function $g_0(z) = \sum_{i=0}^{\infty}(z^i/i!)q_{i0}$. One can show that $g_0(z) = \lim_{t\to\infty} G_0(t;z)$ and that $g_0(z)$ meets the first stationary equation

$$\lambda_2 z(g_0'''(z) - g_0''(z)) + \lambda_1(1 - g_0'(z)) = 0.$$

The solution of the equation has the form

$$g_0(z) = e^z + C\int_0^z \sqrt{x}\, I_1\big(2\sqrt{(\lambda_1/\lambda_2)x}\big)e^{z-x}\,dx, \tag{1}$$

where $I_1(x)$ is the Bessel function and C is the constant. We used the boundary conditions $g_0(0) = 1$, $g_0'(0) = 1$ and $g_0(z)$ is an entire function. The expansion of the expression (1) in a series of powers z and further analysis leads to the following assertion.

Theorem [3]. *The extinction probabilities of the branching process $\xi(t)$ equal,*

$$q_{i0} = 1 - \frac{\Gamma(i-1, \lambda_1/\lambda_2)}{\Gamma(i-1)}, \quad i \geqslant 2.$$

Using the expansion of the incomplete gamma function $\Gamma(i-1, \lambda_1/\lambda_2)$ into a power series (see [6], Chap. V, section C), we obtain the assertion.

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Corollary. *The following representation is valid as* $i \to \infty$,

$$q_{i0} \sim \frac{(\lambda_1/\lambda_2)^{i-1}}{(i-1)!}.$$

For a branching process with pairwise interaction of particles, in which $2T \to k_2T$; $T \to k_1T$, $k_2, k_1 = 0, 1, 2, 3, \ldots$, we have [4], [5] (for some suggestions of general form)

$$q_{i0} \sim C \cdot \frac{q_2^i}{i^\alpha}, \quad i \to \infty,$$

where $0 < q_2 < 1$; here $C > 0$ and α are constants (cf. two particular cases in [2]). In the works [4], [5], authors have considered Laplace transform for the second Kolmogorov equation. In the work [2], we considered stationary first equation.

For an ordinary branching process with independent particles $T \to kT$, $k = 0, 1, 2, \ldots$ we have $q_{i0} = q^i$, $i \in N$, where $0 \leqslant q \leqslant 1$ (see [1], Chap. 2, section 1).

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