ОБОЗРЕНИЕ
ПРИКЛАДНОЙ И ПРОМЫШЛЕННОЙ
Том 23 МАТЕМАТИКИ Выпуск 5
2016

I. G. Shevtsova[1], **V. V. Kudelya**[2] (Lomonosov Moscow State University and Institute for Informatics Problems of FRC «Informatics and Control» of RAS; Lomonosov Moscow State University). **Lower asymptotic estimates in CLT for mean metric.**

For $\delta \in (0,1]$ by $\mathcal{F}_{2+\delta}$ denote a class of all distribution functions (d.f.'s) $F(x)$ satisfying the following conditions

$$\int_{-\infty}^{\infty} x dF(x) = 0, \quad \int_{-\infty}^{\infty} x^2 dF(x) = 1, \quad \beta_{2+\delta} := \int_{-\infty}^{\infty} x^{2+\delta} dF(x) < \infty.$$

Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables with a d. f. $F(x) = \mathsf{P}(X_1 < x)$, $x \in \mathbb{R}$, from $\mathcal{F}_{2+\delta}$. Denote

$$S_n = (X_1 + \cdots + X_n)/\sqrt{n}, \quad F_n(x) = \mathsf{P}(S_n < x), \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

$$\Delta_n = \Delta_n(F) = \int_{-\infty}^{\infty} |F_n(x) - \Phi(x)| dx, \quad L_n^{2+\delta} = L_n^{2+\delta}(F) = \frac{\beta_{2+\delta}}{n^{\delta/2}}.$$

Following the ideas from [4], introduce the asymptotically the best, the exact, the lower asymptotically the best, and the upper asymptotically the best constants:

$$C(\delta) = \sup_{F \in \mathcal{F}_{2+\delta}} \lim_{n\to\infty} \frac{\Delta_n}{L_n^{2+\delta}}, \quad C_0 = \sup_{n\in\mathbb{N}} \sup_{F \in \mathcal{F}_{2+\delta}} \frac{\Delta_n}{L_n^{2+\delta}},$$

$$\underline{C}(\delta) = \limsup_{l\to 0} \limsup_{n\to\infty} \sup_{F \in \mathcal{F}_{2+\delta} : L_n^{2+\delta} = l} \frac{\Delta_n}{L_n^{2+\delta}}, \quad \overline{C}(\delta) = \limsup_{n\to\infty} \sup_{F \in \mathcal{F}_{2+\delta}} \frac{\Delta_n}{L_n^{2+\delta}},$$

respectively. From the definition it trivially follows that $\max\{\underline{C}, C\} \leqslant \overline{C} \leqslant C_0$ for every $\delta \in (0,1]$.

For $\delta = 1$, in 1964 Zolotarev [6] found that $C(1) = 1/2$. In 2009, independently, Tyurin [5] and Goldstein [1] proved an estimate $\Delta_n \leqslant L_n^3$, $n \geqslant 1$, yielding an upper bound $C_0(1) \leqslant 1$. Also, in the same paper [1] Goldstein improved a lower bound for $C_0 \geqslant C \geqslant 1/2$ obtained by Zolotarev to $C_0 \geqslant 0.5353$.

In the following two theorem we prove two lower bounds for $C_0(\delta)$, $\delta \in (0,1]$, in terms of the $\underline{C}(\delta)$ and $\overline{C}(\delta)$.

Theorem 1. *For every* $0 \leqslant \delta \leqslant 1$

$$\underline{C}(\delta) \geqslant \varkappa_{2+\delta}(1+\delta)^{(1+\delta)/2} e^{-(1+\delta)/2}, \tag{1}$$

where

$$\varkappa_{2+\delta} = \frac{\cos\theta_\delta^* - 1 + (\theta_\delta^*)^2/2}{(\theta_\delta^*)^{2+\delta}} = \sup_{x>0} \frac{\cos x - 1 + x^2/2}{x^{2+\delta}},$$

[1]e-mail: ishevtsovacs.msu.ru
[2]e-mail: vitaly.kudelyamail.ru

θ^*_δ *being the unique root of the equation*

$$\frac{\delta\theta^2}{2} + \theta\sin\theta + (2+\delta)(\cos\theta - 1) = 0, \quad 0 < \theta < 2\pi.$$

The values of the lower bounds for $\underline{C}(\delta)$ established by Theorem 1 are given in Table 1 below.

P r o o f. Let $w \in (0, n)$, $s > 0$, and the r.v. X_1 take the values $\pm a$ with probabilities $\gamma_1/2$ and $\pm b$ with probabilities $\gamma_2/2$, where

$$\gamma_1 = \frac{w}{n}, \quad \gamma_2 = 1 - \frac{w}{n}, \quad a = s\sqrt{n}, \quad b = \sqrt{\frac{1-ws^2}{1-w/n}}.$$

Then we have $\mathsf{E}X_1 = 0$, $\mathsf{E}X_1^2 = 1$,

$$\beta_{2+\delta} = \mathsf{E}|X_1|^{2+\delta} = ws^{2+\delta}n^{\delta/2} + (1-ws^2)\left(\frac{1-ws^2}{1-w/n}\right)^{\delta/2},$$

$$L_n^{2+\delta} = \beta_{2+\delta}n^{-\delta/2} = ws^{2+\delta} + O\left(n^{-\delta/2}\right), \quad n\to\infty,$$

so that $\lim\limits_{w\to0}\lim\limits_{n\to\infty} L_n^{2+\delta} = 0$. Thus the $\limsup\limits_{\ell\to0}$ which is taken after the $\limsup\limits_{n\to\infty}$ in the definition of $\underline{C}$ can be replaced by $\lim\limits_{w\to0}$. To estimate Δ_n from below we use the following well-known representation of the mean metric in terms of ζ_1-metric:

$$\Delta_n = \zeta_1(F_n, \Phi) := \sup_f |\mathsf{E}f(S_n) - \mathsf{E}f(Z)|,$$

where Z is a r.v. with the d.f. Φ, and the least upper bound is taken over all bounded functions $f\colon \mathbb{R}\to\mathbb{R}$ such that $|f(x)-f(y)| \leqslant |x-y|$ for every $x, y\in\mathbb{R}$.

Now take any $t > 0$ and fix $f_t(x) = \frac{\cos tx}{t}$. Since $t\mathsf{E}f_t(S_n)$ coincides with the characteristic function of S_n, by independence of $X_1,\ldots,X_n$ we have

$$\mathsf{E}f_t(S_n) = \frac{1}{t}\left(\mathsf{E}\cos\frac{tX_1}{\sqrt{n}}\right)^n = \frac{1}{t}\left(\frac{w}{n}\cos(ts) + \left(1-\frac{w}{n}\right)\cos\left(t\sqrt{\frac{1-ws^2}{n-w}}\right)\right)^n =$$

$$= t^{-1}\exp\left\{w\Big(\cos(ts) - 1 + \frac{t^2s^2}{2}\Big) - \frac{t^2}{2} + o(1)\right\}, \quad n\to\infty.$$

Observing that $\mathsf{E}f_t(Z) = t^{-1}e^{-t^2/2}$, we arrive at the conclusion that for every fixed $s, t, w > 0$ as $n\to\infty$

$$\zeta_1(F_n,\Phi) \geqslant |\mathsf{E}f_t(S_n) - \mathsf{E}f_t(Z)| \longrightarrow t^{-1}e^{-t^2/2}\left|e^{w(\cos(st)-1+s^2t^2/2)} - 1\right| \geqslant$$

$$\geqslant t^{-1}w\Big(\cos(st) - 1 + \frac{s^2t^2}{2}\Big)e^{-t^2/2},$$

and hence,

$$\underline{C}(\delta) \geqslant \frac{(\cos(st) - 1 + \frac{s^2t^2}{2})e^{-t^2/2}}{ts^{2+\delta}}. \tag{2}$$

Using a result of [2] we observe that (2) can be rewritten as (1).

Theorem 2. *For every* $0 < \delta \leqslant 1$

$$\overline{C}(\delta) \geqslant \sup_{\lambda>0}\lambda^{(\delta-1)/2}\Big(\int_{-\infty}^{-\sqrt{\lambda}}\Phi(x)dx + \sum_{k=0}^{\infty}\int_k^{k+1}\left|\sum_{m=0}^{k}\frac{\lambda^m}{m!}e^{-\lambda} - \Phi\Big(\frac{x-\lambda}{\sqrt{\lambda}}\Big)\right|dx\Big). \tag{1}$$

The proof is based of the infinite divisibility of the Poisson distribution and results of [3].

The values of the lower bounds for $\overline{C}(\delta)$ established by Theorem 2 are given in Table below.

Table. The values of the lower bounds for $\underline{C}(\delta)$ and $\overline{C}(\delta)$ from Theorems 1 and 2.

δ	$\underline{C}(\delta) \geqslant$	$\overline{C}(\delta) \geqslant$
0.1	0.2535	0.7053
0.2	0.2139	0.6080
0.3	0.1820	0.5349
0.4	0.1561	0.4775
0.5	0.1349	0.4314
0.6	0.1175	0.3948
0.7	0.1031	0.3669
0.8	0.0912	0.3473
0.9	0.0812	0.3335
1.0	0.0729	0.3249

Acknowledgements. The work is supported by the Russian Foundation for Basic Research (projects 15-07-02984a, 16-31-60110-mol-a-dk) and by the Grant of the President of Russian Federation No. MD–5642.2015.1.

REFERENCES

1. *Goldstein L.* Bound on the constant in the mean central limit theorem. — arXiv:0906.5145, 2009.
2. *Shevtsova I. G.* Moment-type estimates for characteristic functions with application to von Mises inequality. — J. Math. Sci., 2016, v. 214, № 1, p. 119–131.
3. *Shevtsova I. G.* On the accuracy of the normal approximation to compound Poisson distributions. — Theor. Probab. Appl., 2012, v. 58, № 1, p. 138–158.
4. *Shevtsova I. G.* On the asymptotically exact constants in the Berry–Esseen–Katz inequality. — Theor. Probab. Appl., 2011, v. 55, № 2, p. 225–252.
5. *Tyurin I. S.* New estimates of the convergence rate in the Lyapunov theorem. arXiv:0912.0726v1, 2009.
6. *Zolotarev V. M.* On asymptotically best constants in refinements of mean limit theorems. — Theor. Probab. Appl., 1964, v. 9, № 2, p. 268–276.