XXXIII INTERNATIONAL SEMINAR ON STABILITY PROBLEMS FOR STOCHASTIC MODELS

I. V. Zolotukhin[1] (St. Petersburg, P. P. Shirshov Institute of Oceanology RAS, St. Petersburg Department). **Discrete analogue of Marshall–Olkin multivariate exponential distribution.**

Let $\mathcal{E} = \{\boldsymbol{\varepsilon}\}$ is the set of k-dimensional indices $\boldsymbol{\varepsilon} = (\varepsilon^{(1)}, \varepsilon^{(2)}, \ldots, \varepsilon^{(k)})$, each coordinate of which is equal to 0 or 1, and $\mathcal{E}_i$ is the set of indices for which $\varepsilon^{(i)} = 1$.

Let $N_{\boldsymbol{\varepsilon}}$ are the independent geometrically distributed random variables:

$$\mathbf{P}\{N_{\boldsymbol{\varepsilon}} = n\} = p_{\boldsymbol{\varepsilon}}\, q_{\boldsymbol{\varepsilon}}^{n-1}, \quad n = 1, 2, \ldots, \quad p_{\boldsymbol{\varepsilon}} = 1 - q_{\boldsymbol{\varepsilon}}$$

with the characteristic functions

$$\varphi_{\boldsymbol{\varepsilon}}(t) = \frac{p_{\boldsymbol{\varepsilon}} e^{it}}{1 - q_{\boldsymbol{\varepsilon}}\, e^{i\,t}}$$

The class of these distributions will be denoted by $G(q_{\boldsymbol{\varepsilon}})$.

Let's take into consideration the random variables

$$M_i = \min_{\boldsymbol{\varepsilon} \in \mathcal{E}_i} \{N_{\boldsymbol{\varepsilon}}\}, \quad i = 1, 2, \ldots, k.$$

Let $N_{\boldsymbol{\varepsilon}} = \infty$ if $p_{\boldsymbol{\varepsilon}} = 0$.

Definition. Distribution of the vector

$$\mathbf{M} = (M_1, M_2, \ldots, M_k) = (\min_{\boldsymbol{\varepsilon} \in \mathcal{E}_1} \{N_{\boldsymbol{\varepsilon}}\}, \min_{\boldsymbol{\varepsilon} \in \mathcal{E}_2} \{N_{\boldsymbol{\varepsilon}}\}, \ldots, \min_{\boldsymbol{\varepsilon} \in \mathcal{E}_k} \{N_{\boldsymbol{\varepsilon}}\})$$

is called *multivariate geometric distribution*, and the vector $\mathbf{M}$ is called *geometrically distributed random vector*.

Notation. Let us denote $MVG(q_{\boldsymbol{\varepsilon}}, \boldsymbol{\varepsilon} \in \mathcal{E})$ the class of multivariate geometric distributions.

In what follows the vector $\boldsymbol{\varepsilon}$ will be used for the indication of the coordinate hyperplane in the k-dimensional space, $\mathbf{0} = (0, 0, \ldots, 0)$, $\mathbf{1} = (1, 1, \ldots, 1)$, $\varepsilon = \mathbf{1} - \boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}_1 \vee \boldsymbol{\varepsilon}_2$ is the vector, each ith coordinate is $\max\{\varepsilon_1^{(i)}, \varepsilon_2^{(i)}\}$.

Define the partial order $\leqslant$ on the set $\mathcal{E}$ by the rule:

$$(\forall \boldsymbol{\varepsilon}, \boldsymbol{\delta} \in \mathcal{E}) \quad \boldsymbol{\delta} \leqslant \boldsymbol{\varepsilon}, \text{ if } (\forall j \in \{1, 2, \ldots, k\} \quad \boldsymbol{\delta}^{(j)} \leqslant \varepsilon^{(j)}.$$

Let $\mathbf{m} = (m_1, m_2, \ldots, m_k)$, m_j are the natural numbers, $\boldsymbol{\varepsilon}\mathbf{m}$ will mean their coordinate-wise product. Denotation $(\mathbf{a}, \mathbf{b})$ means the scalar product of vectors $\mathbf{a}$ и $\mathbf{b}$.

Theorem 1. *The projection $\boldsymbol{\varepsilon}\mathbf{M}$ of the vector $\mathbf{M}$ on the coordinate hyperplane $\boldsymbol{\varepsilon}$ has a multivariate geometric distribution with the reliability function*

$$\overline{P}(\boldsymbol{\varepsilon}\mathbf{m}) = \prod_{\boldsymbol{\delta} \leqslant \boldsymbol{\varepsilon}} \Bigg(\prod_{\boldsymbol{\gamma} : \boldsymbol{\gamma}\boldsymbol{\varepsilon} = \boldsymbol{\delta}} q_{\boldsymbol{\gamma}} \Bigg)^{\max_{1 \leqslant i \leqslant k} \boldsymbol{\delta}\mathbf{m}}.$$

Theorem 2. *The characteristic function of the multivariate geometric distribution is*

$$\psi(\mathbf{t}) = \mathbf{E}\, e^{i\,(\mathbf{t}, \mathbf{M})} = \frac{e^{i\,(\mathbf{t}, \mathbf{1})}}{1 - e^{i\,(\mathbf{t}, \mathbf{1})} \prod_{\boldsymbol{\delta} \in \mathcal{E}} q_{\boldsymbol{\delta}}} \sum_{j=1}^{k} \sum_{\substack{\boldsymbol{\delta}_l \in \mathcal{E} \\ l = 1, 2, \ldots, j}} \prod_{l=1}^{j} p_{\boldsymbol{\delta}_l} \prod_{\substack{\boldsymbol{\gamma} \neq \boldsymbol{\delta}_l \\ l = 1, 2, \ldots, j}} q_{\boldsymbol{\gamma}}\, \psi\Big(\overline{\bigvee\nolimits_{l=1}^{j} \boldsymbol{\delta}_l \mathbf{t}} \Big).$$

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The summation in the second sum is carried over all values of $\boldsymbol{\delta}_l$, differing among themselves.

Theorem 3. *The characteristic function of the projection of vector* $\mathbf{M}$ *on hyperplane* $\boldsymbol{\varepsilon}$ *has the form*

$$\psi(\boldsymbol{\varepsilon}\mathbf{t}) = \mathbf{E}\, e^{\,i\,(\mathbf{t},\boldsymbol{\varepsilon}\mathbf{M})} = \frac{e^{\,i\,(\mathbf{t},\boldsymbol{\varepsilon})}}{1-e^{\,i\,(\mathbf{t},\boldsymbol{\varepsilon})}\prod_{\boldsymbol{\delta}:\boldsymbol{\delta}\,\boldsymbol{\varepsilon}>\mathbf{0}} q_{\boldsymbol{\delta}}} \sum_{j=1}^{k} I_{j,\boldsymbol{\varepsilon}},$$

where

$$I_{j,\boldsymbol{\varepsilon}} = \sum_{\substack{\boldsymbol{\delta}_l:\boldsymbol{\delta}_l\boldsymbol{\varepsilon}>\mathbf{0}\\ l=1,2,\ldots,j}} \prod_{l=1}^{j} p_{\boldsymbol{\delta}_l} \prod_{\substack{\boldsymbol{\gamma}:\boldsymbol{\gamma}\,\boldsymbol{\varepsilon}>\mathbf{0}\\ \boldsymbol{\gamma}\neq\boldsymbol{\delta}_l\\ l=1,2,\ldots,j}} q_{\boldsymbol{\gamma}}\,\psi\Big(\overline{\bigvee_{l=1}^{j}\boldsymbol{\delta}_l\,\boldsymbol{\varepsilon}\,\mathbf{t}}\Big), \quad \mathbf{t}=(t_1,t_2,\ldots,t_k).$$

Theorem 4. *Let* $\mathbf{M}\in MVG(q_{\boldsymbol{\varepsilon}},\boldsymbol{\varepsilon}\in\mathcal{E})$ *and* $p_{\boldsymbol{\varepsilon}}=\lambda_{\boldsymbol{\varepsilon}}p$, *then* $p\,\mathbf{M}\xrightarrow[p\to 0]{\mathcal{D}}\mathbf{V}$, *where* $\mathbf{V}$ *has the Marshall–Olkin multivariate exponential distribution.*

REFERENCES

Marshall A. W., Olkin I. A. Multivariate exponential distribution. — J. Amer. Statist. Assoc., 1967, v 62, № 317, p. 30–44.