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**M. Al–N a t o r** [1] (Financial University, Moscow). **Portfolio analysis with general commission.**

In this work we generalize the results of [1] (see also [2]). Namely the well-known problem of finding explicit formulas for the expected return and risk of portfolios with general commission is completely solved. It is assumed that the commission depends on the asset and the asset position, and on whether the given position is opened or closed. For portfolios with only the budget constraint and initial commission we prove that the function of expected portfolio return and portfolio variance are bounded.

Let us introduce some notation. Suppose that we have $n$ assets $A_1, A_2, \ldots, A_n$. Let $R_k$ (respectively, $r_k = E(R_k)$ ) denotes the random price return (respectively, expected price return) of $A_k$. The portfolio will be denoted by the vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ of asset weights such that $x_1 + x_2 + \cdots + x_n = 1$ (the budget constraint). If there are no commission costs, it is well known [3–6] that the portfolio return $R(\boldsymbol{x})$ (respectively, the expected return $r(\boldsymbol{x})$ ) is the weighted average of the individual asset returns (respectively, the expected returns):

$$R(\boldsymbol{x}) = \langle \boldsymbol{R}, \boldsymbol{x} \rangle, \quad r(\boldsymbol{x}) = \langle \boldsymbol{r}, \boldsymbol{x} \rangle$$

where $\boldsymbol{R} = (R_1, R_2, \ldots, R_n)$, $\boldsymbol{r} = (r_1, r_2, \ldots, r_n)$, and $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbf{R}^n$. It follows that the portfolio variance of return or risk is given by $V(\boldsymbol{x}) = \langle C\boldsymbol{x}, \boldsymbol{x} \rangle$, where $C$ is the covariance matrix of asset returns: $c_{ij} = \operatorname{cov}(R_i, R_j)$.

Denote by $\boldsymbol{\alpha}^{\boldsymbol{\varepsilon}} = (\alpha_1^{\varepsilon_1}, \alpha_2^{\varepsilon_2}, \ldots, \alpha_n^{\varepsilon_n})$ (respectively, $\boldsymbol{\beta}^{\boldsymbol{\varepsilon}} = (\beta_1^{\varepsilon_1}, \beta_2^{\varepsilon_2}, \ldots, \beta_n^{\varepsilon_n})$ ) the vector of initial ((respectively, final) commission, where $\alpha_k^{\varepsilon_k}$ (respectively, $\beta_k^{\varepsilon_k}$ ) is the commission for opening (respectively, closing) the position of the asset $A_k$ and $\varepsilon_k = \pm$. Here $\varepsilon_k = +$ if the investor opens or closes a long $k$-th position and $\varepsilon_k = -$ if the investor opens or closes a short $k$-th position.

Consider the portfolio $\boldsymbol{x}$ with commissions $\boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}$, $\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}$.

**Theorem 1.** *Let $R_{\boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}(\boldsymbol{x})$ be the portfolio return and let $r_{\boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}(\boldsymbol{x})$ be the expected portfolio return. Then*

$$R_{\boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}(\boldsymbol{x}) = \frac{\langle \boldsymbol{R}, \boldsymbol{x} \rangle - \langle \boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+ \rangle - \langle \boldsymbol{\beta}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+^{\boldsymbol{a}} \rangle}{1 + \langle \boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+ \rangle} = \frac{\langle \overline{\boldsymbol{x}}^{\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}, \boldsymbol{R} \rangle - \langle \boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+ \rangle}{1 + \langle \boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+ \rangle},$$

$$r_{\boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}(\boldsymbol{x}) = \frac{\langle \boldsymbol{r}, \boldsymbol{x} \rangle - \langle \boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+ \rangle - \langle \boldsymbol{\beta}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+^{\boldsymbol{b}} \rangle}{1 + \langle \boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+ \rangle} = \frac{\langle \overline{\boldsymbol{x}}^{\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}, \boldsymbol{r} \rangle - \langle \boldsymbol{\gamma}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+ \rangle}{1 + \langle \boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+ \rangle},$$

*where* $\boldsymbol{x}_+ = (|x_1|, |x_2|, \ldots, |x_n|)$, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$, $a_k = 1 + R_k$, $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$, $b_k = 1 + r_k$, $\boldsymbol{x}_+^{\boldsymbol{a}} = (a_1|x_1|, a_2|x_2|, \ldots, a_n|x_n|)$, $\boldsymbol{x}_+^{\boldsymbol{b}} = (b_1|x_1|, b_2|x_2|, \ldots, b_n|x_n|)$, $\overline{\boldsymbol{x}}^{\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}} = \boldsymbol{x} - \boldsymbol{x}_+^{\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}$, $\boldsymbol{x}_+^{\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}} = (\beta_1^{\varepsilon_1}|x_1|, \beta_2^{\varepsilon_2}|x_2|, \ldots, \beta_n^{\varepsilon_n}|x_n|)$.

*In particular, the portfolio risk $V_{\boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}(\boldsymbol{x})$ is equal to*

$$V_{\boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}(\boldsymbol{x}) = \frac{\langle C\overline{\boldsymbol{x}}^{\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}, \overline{\boldsymbol{x}}^{\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}} \rangle}{(1 + \langle \boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+ \rangle)^2} = \frac{\langle C\boldsymbol{x}, \boldsymbol{x} \rangle - 2\langle C\boldsymbol{x}, \boldsymbol{x}_+^{\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}} \rangle + \langle C\boldsymbol{x}_+^{\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}}, \boldsymbol{x}_+^{\boldsymbol{\beta}^{\boldsymbol{\varepsilon}}} \rangle}{(1 + \langle \boldsymbol{\alpha}^{\boldsymbol{\varepsilon}}, \boldsymbol{x}_+ \rangle)^2}$$


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Note that $R_{\mathbf{0},\mathbf{0}}(\boldsymbol{x}) = R(\boldsymbol{x})$, $r_{\mathbf{0},\mathbf{0}}(\boldsymbol{x}) = r(\boldsymbol{x})$ and $V_{\mathbf{0},\mathbf{0}}(\boldsymbol{x}) = V(\boldsymbol{x})$.

For $\boldsymbol{z} = (z_1, z_2, \ldots, z_n) \in \mathbf{R}^n$ we set $||\boldsymbol{z}|| = |z_1|+|z_2|+\cdots+|z_n|$, $\boldsymbol{z}_{\min} = \min_{1\leqslant i\leqslant n}\{z_i\}$, $\boldsymbol{z}_{\max} = \max_{1\leqslant i\leqslant n}\{z_i\}$, $\boldsymbol{\gamma}^{\varepsilon} = \boldsymbol{\alpha}^{\varepsilon} + \boldsymbol{\beta}^{\varepsilon}$.

**Theorem 2.** *For fixed* $\boldsymbol{\alpha}^{\varepsilon} \neq \mathbf{0}$ *and* $\boldsymbol{\beta}^{\varepsilon}$ *the functions* $r_{\boldsymbol{\alpha}^{\varepsilon},\boldsymbol{\beta}^{\varepsilon}}(\boldsymbol{x})$, $V_{\boldsymbol{\alpha}^{\varepsilon},\boldsymbol{\beta}^{\varepsilon}}(\boldsymbol{x})$ *are bounded, namely*

$$|r_{\boldsymbol{\alpha}^{\varepsilon},\boldsymbol{\beta}^{\varepsilon}}(\boldsymbol{x})| < \frac{n\,(1+\boldsymbol{\beta}^{\varepsilon}_{\max})\,||\boldsymbol{r}|| + \boldsymbol{\gamma}^{\varepsilon}_{\max}}{\boldsymbol{\alpha}^{\varepsilon}_{\min}}, \quad 0 \leqslant V_{\boldsymbol{\alpha}^{\varepsilon},\boldsymbol{\beta}^{\varepsilon}}(\boldsymbol{x}) < n\Big(\frac{1+\boldsymbol{\beta}^{\varepsilon}_{\max}}{\boldsymbol{\alpha}^{\varepsilon}_{\min}}\Big)^2 \max_{1\leqslant i,j\leqslant n} |c_{ij}|.$$

Under some restrictions on the commission one can obtain the exact boundaries for $r_{\boldsymbol{\alpha}^{\varepsilon},\boldsymbol{\beta}^{\varepsilon}}(\boldsymbol{x})$, $V_{\boldsymbol{\alpha}^{\varepsilon},\boldsymbol{\beta}^{\varepsilon}}(\boldsymbol{x})$.

## REFERENCES


1. *Al-Nator M. S.*, *Al-Nator S. V.*, *Kasimov Yu. F.* Portfolio analysis with transaction costs under uncertainty. — J. Math. Sci., 2016, v. 214, № 1, p. 12–21.
2. *Al-Nator M.*, *Al-Nator S.*, *Kasimov Yu.* Portfolio analysis with transaction costs under uncertainty. — In: XXXII International Seminar on Stability Problems for Stochastic Models and VIII International Workshop “Applied Problems in Theory of Probabilities and Mathematical Statistics Related to Modeling of Informations Systems” (Trondheim, Norway, June 16–21, 2014.). Book of Abstracts. / Ed. by V. Yu. Korolev, S. Ya. Shorgin. Moscow: IIP RAS, p. 14–16.
3. *Luenberger D. G.* Investment Science. 2nd ed. N. Y.: Oxford Univ. Press Inc, 2013, 640 p.
4. *Sharpe W.*, *Alexander G. J.*, *Bailey J. W.* Investments. 6th ed. Upper Saddler River, NJ: Prentice-Hall, 1999, 962 p.
5. *Al-Nator M. S.*, *Kasimov Yu. F.*, *Kolesnikov A. N.* Fundamentals of Financial Calculations (Facts, Formulas, Cases and Problems). Part 2. Moscow: Financial Univ. Publ., 2013, 185 p. (In Russian.)
6. *Al-Nator M. S.*, *Kasimov Yu. F.*, *Kolesnikov A. N.* Fundamentals of Financial Calculations (Facts, Formulas, Cases, Problems aand Tests). Part 3. Moscow: Financial Univ. Publ., 2014, 149 p. (In Russian.)