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A. L. Talis, A. L. Rabinovich (Moscow, INEOS RAS; Petrozavodsk, IB KarRC RAS). **Hyperbolic tetrahedral honeycomb, symmetry of the Klein quartic and the hidden (non-crystallographic) symmetry of components of natural phospholipids.**

Hydrocarbon components of natural phospholipids are linear chains; they can be approximated by chains of identical regular tetrahedra (which allow obtaining tetracoordinate structures). To reveal the "hidden" symmetry of such chain molecules in 3-dimensional Euclidean space E^3, an important role can be played by the symmetry of their analogs in 3-dimensional Riemannian spaces: according to [1, p. 99], there is an osculating Euclidean space to the given Riemannian space along the length of a given line. Let us consider the case of the space H^3 of constant negative curvature, tesselated by ideal tetrahedra; these are hyperbolic honeycomb $\{3,3,6\}$, each edge of which has 6 ideal hyperbolic tetrahedra [2, 3]. It is necessary to reveal the maximal basic structural unit, which is the join of face-sharing regular tetrahedra, and the number of vertices of which does not change under mapping from H^3 to E^3. Such a unit can not keep more than 5 tetrahedra because it's impossible to combine 6 regular tetrahedra at one edge in E^3. There are only 8 finite-volume universal regular hyperbolic tetrahedral tesselations [2], and the desired unit must have the ability of embedding to each of them. The only compromise option is the 7-vertex linear join of 4 face-sharing regular tetrahedra, the tetrablock.

According to [4], the "right-hand" join $M(T_{2+\zeta})$ containing 28 hyperbolic tetrahedra (where $T_{2+\zeta}$ is the 7-vertex triangulation of a torus,

$$\zeta = \frac{1+\sqrt{-3}}{2}, \quad (2+\zeta)(2+\bar{\zeta}) = 7$$

[2]) is embedded in the honeycomb $\{3,3,6\}$. It has a symmetry group PSL(2,7) of order 168, and is one of the above mentioned 8 finite-volume universal regular hyperbolic tetrahedral tesselations. It is shown that there is a possibility to obtain the $M(T_{2+\zeta})$ tesselation as 7 quadruples of hyperbolic tetrahedra (which correspond to tetrablocks), and all the sets considered in [4] can be defined by decompositions into the adjacent classes of group $\mathrm{PSL}(2,7) \equiv {}^7O$, — that is the Klein quartic symmetry group, namely

$${}^7O = \bigcup_{i=1}^{56} g_i C_3 = \bigcup_{n=1}^{28} g_n D_3 = \bigcup_{j=1}^{24} g_j C_7 = \bigcup_{k=1}^{8} g_k M_{3,7} = \bigcup_{m=1}^{7} g_m O' = \bigcup_{f=1}^{4} C_2 g_f O',$$

where C_3 and C_7 are the symmetry groups of the triangle and the heptagon, $C_3 \subset D_3$, D_3 is dihedral group, $M_{3,7}$ is the direct product of C_3 and C_7, the subgroups O' and O'' (the "left-hand" join $M(T_{2+\bar{\zeta}})$ of the 28 tetrahedra corresponds to the above decompositions of ${}^7O$ with the group O'') are not conjugated in ${}^7O$ and coincide with the rotation group of the cube, C_2 is the symmetry group of the "right-hand" tetrablock,

$$g_i \notin C_3, \quad g_n \notin D_3, \quad g_j \notin C_7, \quad g_k \notin M_{3,7}, \quad g_m, g_f \notin O'.$$

The numbers of cosets 56, 24, 8, 7 are determined by the fact that in the Klein quartic [2, 4] (that is defined as the tesselation of a sphere with 3 handles into 24 regular hyperbolic heptagons) 3 heptagons meet at each of the 56 vertices belonging to 7 cubes. When the centers of all 24 heptagons are joined, then 56 equilateral hyperbolic triangles are formed, each of degree 7 meeting at 24 vertices. Each of the 8 cyclic subgroups C_7 ("axes" C_7) of the group ${}^7O$ fixes a certain triple of heptagons. The amphichiral tetrablock also exists, it is determined by the decomposition of

$$\mathrm{PGL}(2,7) \equiv {}^7O_h = \bigcup_{n=1}^{4} C_{2v} g_n O_h,$$

where O_h is the point group of cube, $C_2 \subset C_{2v}$, $C_{2v} \not\ni g_n \notin O_h$. The existence of the tetrablock symmetry is the necessary, but not sufficient condition for the formation of real structures in phospholipid components (similar to the existence of the Fedorov group, which determines the possibility of crystalline ordering, and the real crystal).

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