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A. L. Rabinovich, **A. L. Talis** (Petrozavodsk, IB KarRC RAS; Moscow, INEOS RAS). **Combinatorial constructions and non-crystallographic symmetry of tetrahedral and tetra-coordinate structures.**

Rectilinear rods of face-sharing tetrahedra have not translational symmetry but distortions of the tetrahedra allow a translational repeat. The minimal distortions correspond to the periodic set of 8 tetrahedra [1, 2]; and such a distorted rod was used to describe some real crystal structures [1]. To elucidate the cause of such regularities let's consider a series of combinatorial constructions which can be related to the linear regular face-sharing tetrahedra; a diamond-like join of two such tetrahedral chains results in a tetra-coordinate structure (e. g., a hydrocarbon chain). The set $\{1,2,3,4\}$ of a tetrahedron vertices is divided into 4 triples (faces):

$$\{123\}, \quad \{134\}, \quad \{124\}, \quad \{234\};$$

any pair of the vertices (i. e. an edge) belongs solely to 2 triples (i.e. faces). The combinatorial structure of a regular tetrahedron can be considered as a special case of a t-(v,k,λ) design [3, p. 88]: a v-set with $v=4$ elements as a collection of distinct k-subsets (called *blocks*) of $k=3$ elements with the property that any t-subset of $t=2$ elements is contained in exactly λ blocks ($\lambda=2$). The parameters satisfy the following condition:

$$v = 1 + \frac{k\,(k-1)}{2}\,.$$

The variants of a t-(v,k,λ) design at $t=\lambda=2$, i. e., 2-$(v,k,2)$ designs (which correspond to polyhedra) are all that are potentially important for us. Regular tetrahedron vertices ($v=4$) belong to a regular triangle tessellation of a sphere. In the general case totally symmetric regular triangulation of a surface is equivalent to the triangular embedding of the complete graph into "the orientable surface of genus p" (i.e. the sphere with p "handles" on it) [4]. It happens only if the number of vertices [4, p. 74]

$$v = 0,\ 3,\ 4,\ 7 \mod 12.$$

Taking into account that 2-$(v,k,2)$ designs can be realized only if $v=4,7,11,\ldots$, the coincidence in v values (if not in the face shape) is reached only if $v=4$ and 7. It is possible to realize the regular triangulation at $v=7$ only if the surface is a torus [4; p. 74] (the sphere with 1 handle, $p=1$): the tessellation contains 14 triangles, 21 edges. Combinatorial construction symmetry is represented by an incidence table (IT) where the elements belonging to each block are indicated. No "triangular" geometric interpretation exists for 2-(7,4,2) design but IT for 2-(7,4,2) design is complementary to IT for 2-(7,3,1) design, and the latter has the only interpretation: the 7-vertices regular triangulation of a torus [5]. The symmetry group of both 2-(7,4,2) and 2-(7,3,1) designs is PSL(2,7) [6]. In this case, the IT for the 2-(7,3,1) design, in which the incidence signs are equivalent, coincides with the IT for the minimal finite projective plane PG(2,2) [5], in which the points differ (7 points in all, 3 pairs of parallel lines intersect at 3 points lying at infinity). It allows distinguishing [7] the incidence signs: for each individual line section in PG(2,2), the incidence sign in the IT of PG(2,2) can be identified. To convert the triangulated torus to the tetrahedral structure in the 3-dimensional Euclidean space E^3, we excluded from the IT of PG(2,2) precisely 3 incidence signs which are responsible in the PG(2,2) for the intersection of parallel lines. The resulting Euclidean sub-configuration of the PG(2,2) defies 4 face-sharing tetrahedra, a tetra-block (7 vertices, 10 triangles, 15 edges). It is shown that the tetra-block symmetry group is isomorphic to the PSL(2,7) group (its order is 168 [6]). A linear face-to-face join of 2 tetra-blocks (11 vertices) corresponds to the set of 8 tetrahedra and has the symmetry group that is isomorphic to the group PSL(2,11) (its order is 660 [6]).

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