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In this note we are concerned with theorems to the real valued periodic functions of n real arguments defined on all of $\mathbb{R}^n$. A function $f:\mathbb{R}^n\to\mathbb{R}$ is called periodic with period $\overline{T}$ if there exists a vector $\overline{T}\neq\overline{0}$ such that $f(\overline{r}+\overline{T})=f(\overline{r})$ for all $\overline{r}\in\mathbb{R}^n$. This concept is used in the mathematical modeling of self-similar objects and their properties, and of different repetitive processes in time and space. For example, it arises in the study of the band structure of solid state [1]. The wave function ψ satisfies the Born–Karman conditions $\psi(\overline{r}+N_i\overline{a}_i)=\psi(\overline{r})$, $i=1,\ldots,d$, where d is dimension of Bravais lattice, $\overline{a}_i$ are its primitive vectors, N_i are integers. We study the relationship between the periodicity of a multivariate function in the sense of the above definition and its periodicity with respect to individual variables.

Definition 1. Let $f:\mathbb{R}^n\to\mathbb{R}$ be a periodic function with period $\overline{T}$. If there exists a period $\overline{T}_0$ of the least magnitude and direction of the vector $\overline{T}$, then it is called a basic period of function f along a given unit vector $\overline{\mathcal{T}}$, where $\overline{T}=|\overline{T}|\cdot\overline{\mathcal{T}}$.

Now suppose the set of a straight lines $\ell_{\overline{\mathcal{T}}}(\overline{a})$ in $\mathbb{R}^n$ parallel to the vector $\overline{T}$, where $\overline{a}$ is a radius vector of certain point of $\ell_{\overline{\mathcal{T}}}(\overline{a})$. Let us choose this point such that $\langle\overline{a},\overline{T}\rangle=0$, then the correspondence $\overline{a}\to\ell_{\overline{\mathcal{T}}}(\overline{a})$ is one-to-one. Restriction of the function $f:\mathbb{R}^n\to\mathbb{R}$ to any one of the considered straight lines is given by

$$f(\overline{r})\big|_{\overline{r}\in\ell_{\overline{\mathcal{T}}}(\overline{a})}=f(\overline{a}+t\overline{\mathcal{T}}),\qquad(1)$$

it is a function of one variable $t\in\mathbb{R}$.

Theorem 1. *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a periodic function with period $\overline{T}$. If at least one of the restrictions of function f to the straight lines $\ell_{\overline{\mathcal{T}}}(\overline{a})$ is continuous and nonconstant, then there exists basic period of f along a given unit vector $\overline{\mathcal{T}}$.*

Theorem 2. *If $f:\mathbb{R}^n\to\mathbb{R}$ is periodic function with basic period $\overline{T}_0$ along a given unit vector $\overline{\mathcal{T}}$, then any its period $\overline{T}$ parallel to the vector $\overline{\mathcal{T}}$ is $\overline{T}=k\cdot\overline{T}_0$, where $k\in\mathbb{Z}\setminus\{0\}$.*

Theorem 3. *Suppose that $f:\mathbb{R}^n\to\mathbb{R}$ is periodic function with basic period $\overline{T}_0$ along a given unit vector $\overline{\mathcal{T}}$, and $\mathcal{A}:\mathbb{R}^n\to\mathbb{R}^n$ is nonsingular linear operator. Then the function composition $f\circ\mathcal{A}:\mathbb{R}^n\to\mathbb{R}$ is periodic with basic period $\mathcal{A}^{-1}\overline{T}_0$ along a respective unit vector $\overline{\tau}$, where $\mathcal{A}^{-1}\overline{T}_0=|\mathcal{A}^{-1}\overline{T}_0|\cdot\overline{\tau}$.*

P r o o f. Let $f:\mathbb{R}^n\to\mathbb{R}$ be a periodic function with basic period $\overline{T}_0$; then it follows from the equalities $f(\mathcal{A}(\overline{r}+\mathcal{A}^{-1}\overline{T}_0))=f(\mathcal{A}\overline{r}+\overline{T}_0)=f(\mathcal{A}\overline{r})$ for all $\overline{r}\in\mathbb{R}^n$ that the vector $\mathcal{A}^{-1}\overline{T}_0$ is a period of function composition $f\circ\mathcal{A}:\mathbb{R}^n\to\mathbb{R}$. Let us suppose that $\mathcal{A}^{-1}\overline{T}_0$ is not basic period of this function along a given unit vector $\overline{\tau}$, then there exists another period $\overline{T}$ of the least magnitude and parallel to the vector $\overline{\tau}$, i. e. $\overline{T}=\lambda\mathcal{A}^{-1}\overline{T}_0$, where $0<\lambda<1$, and $f(\mathcal{A}(\overline{r}+\overline{T}))=f(\mathcal{A}\overline{r})$. On the other hand, the vector $\mathcal{A}\overline{T}=\lambda\overline{T}_0$

is a period of function f, and $|\mathcal{A}\overline{T}| < |\overline{T}_0|$. This fact contradicts our previous assumption that $\overline{T}_0$ is basic period of function f along a given unit vector $\overline{\mathcal{T}}$.

R e m a r k 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a periodic function, the vector $\overline{T}_0$ is its basic period along a given unit vector $\overline{\mathcal{T}}$, and $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$ is nonsingular linear operator. If the components of the vector $\overline{\mathcal{T}}$ are elements of the i-th column of the matrix A of linear operator $\mathcal{A}$, then the function composition $f \circ \mathcal{A} : \mathbb{R}^n \to \mathbb{R}$ is a periodic with basic period $|\overline{T}_0|\overline{e}_i$ along a given unit vector $\overline{e}_i$. In other words, this function is a periodic with respect to variable x_i, and number $|\overline{T}_0|$ is its basic period. Here and everywhere below $\{\overline{e}_i\}_{i=1}^n$ is a natural Hamel basis in $\mathbb{R}^n$.

R e m a r k 2. It is possible for a m-dimensional lattice [3, p. 11] to be a set of periods of a periodic function $f : \mathbb{R}^n \to \mathbb{R}$. A set of all possible non-trivial linear combinations of m linearly independent n-dimensional vectors $\overline{T}_1, \overline{T}_2, \ldots, \overline{T}_m$ with integers as coefficients is called m-dimensional lattice. The vectors $\overline{T}_1, \overline{T}_2, \ldots, \overline{T}_m$ are called a primitive vectors of the lattice, they are necessarily basic periods of function f in respective directions. By choosing the matrix A of linear operator $\mathcal{A}$ it is possible for function $f : \mathbb{R}^n \to \mathbb{R}$ to be a periodic with respect to any m variables with periods $|\overline{T}_1|, |\overline{T}_2|, \ldots, |\overline{T}_m|$ respectively.

If the restriction of the function $f : \mathbb{R}^n \to \mathbb{R}$ to any straight line $\ell_{\overline{\mathcal{T}}}(\overline{a})$ is constant, then this function is called a constant along a given unit vector $\overline{\mathcal{T}}$. It means that for any fixed vector $\overline{a}$ the function (1) does not depend in variable t.

Theorem 4. *If $f : \mathbb{R}^n \to \mathbb{R}$ is constant along a given unit vector $\overline{\mathcal{T}}$, then it is periodic function with period $\alpha\overline{\mathcal{T}}$, where $\alpha \in \mathbb{R} \setminus \{0\}$.*

R e m a r k 3. If the function $f : \mathbb{R}^n \to \mathbb{R}$ is constant along all linearly independent given unit vectors $\overline{\mathcal{T}}_1, \overline{\mathcal{T}}_2, \ldots, \overline{\mathcal{T}}_k$, then the linear span of the vectors $\overline{\mathcal{T}}_1, \overline{\mathcal{T}}_2, \ldots, \overline{\mathcal{T}}_k$ is a set of periods of this function. Here $k \leqslant n$, in the case of $k = n$ the function f is identically constant.

R e m a r k 4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a constant function along a given unit vector $\overline{\mathcal{T}}$. If the components of the vector $\overline{\mathcal{T}}$ are elements of the ith column of the matrix A of linear operator $\mathcal{A}$, then the function composition $f \circ \mathcal{A} : \mathbb{R}^n \to \mathbb{R}$ is constant with respect to variable x_i, i. e. this function does not depend in variable x_i.

R e m a r k 5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a constant function along all linearly independent given unit vectors $\overline{\mathcal{T}}_1, \overline{\mathcal{T}}_2, \ldots, \overline{\mathcal{T}}_k$, where $k \leqslant n$. By choosing the matrix A of linear operator $\mathcal{A}$ it is possible for function $f \circ \mathcal{A} : \mathbb{R}^n \to \mathbb{R}$ to be a constant with respect to chosen of k variables.

Without loss of generality it can be assumed that any function $f : \mathbb{R}^n \to \mathbb{R}$ is a periodic with respect to m variables, it is constant with respect to other k variables, and it is non-periodic with respect to $n - m - k$ remaining variables.

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