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V. P. Zhitnikov, **N. M. Sherykhalina**, **R. R. Muksimova** (Ufa, USATU, Saint-Petersburg, SGbGU GA). **Accumulation of error in solving of a mixed problem for the one-dimensional heat conduction equation by the iterative method.**

The investigation of the process of error accumulation caused by the inaccuracy of the systems of linear algebraic equations (SLAE) solution by the iterative method in several time steps is carried out. The two types of errors are usually considered in problems solving by finite-difference methods: the theoretical error of the finite-difference method and the round off error [1]. However, when iterative methods for solving systems of linear and nonlinear equations are applied [2], the third kind of error arises: the accumulated error caused by the inaccuracy of the equations systems solving at each time step.

Let us consider a rod made of heat-conducting material, the temperature at the ends of the rod is specified, and the side surface of the rod is thermally isolated. The ends of the rod are located at the points $x = 0$ and $x = 1$. The problem is reduced to determining of the dependence of the temperature on time and the defining of the temperature coordinate a t the points of the rod $u(x, t)$, which must satisfy the heat conduction equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$, ($0 < x < 1$, $0 < t \leqslant T$) and, for example, the following initial and boundary conditions

$$f(x) = \sin(\pi x), \quad \varphi_1(t) = \varphi_2(t) = 0, \tag{1}$$

with the accurate solution

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x). \tag{2}$$

The problem is solved by the method of finite differences

$$u_{i,j} = \frac{\lambda}{1+2\lambda} u_{i-1,j} + \frac{\lambda}{1+2\lambda} u_{i+1,j} + \frac{u_{i,j-1}}{1+2\lambda}, \quad u_{i,j} = u(x_i, t_j), \quad \lambda = \tau/h^2,$$

(τ, h are steps in time and space). The SLAE solution can be carried out by the simple iteration method. The iterative process $x^{(k)} = Bx^{(k-1)} + b$ converges for any λ with the rate of geometric progression

$$\left\| x^{(k)} - x \right\| \leqslant \|B\| \left\| x^{(k-1)} - x \right\|, \quad \|x\| = \max_{0 \leqslant i \leqslant n} |x_i|, \quad \|B\| = \frac{2\lambda}{1+2\lambda}. \tag{3}$$

The condition for the completion of the iterative process is inequality

$$\left\| x^{(k)} - x^{(k-1)} \right\| \leqslant \varepsilon. \tag{4}$$

Using (4) and the absolute error estimate (3), we obtain at the j-th time step

$$\Delta_j = \left\| x^{(k)} - x \right\| \leqslant \frac{\|B\|}{1 - \|B\|} \left\| x^{(k)} - x^{(k-1)} \right\| \leqslant 2\lambda\varepsilon.$$

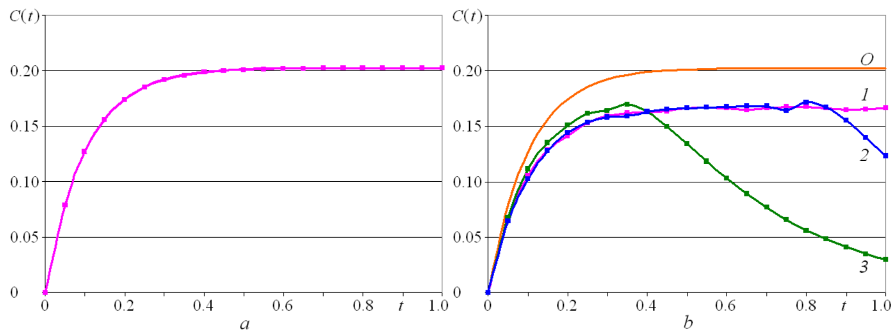

Figure 1. The dependence of coefficient of absolute error on t for $n = 640$: $a - m = 2n$; $b - \lambda = 1$

Because of $\lambda = \tau/h^2 = n^2\tau$, then

$$\Delta_j \leqslant 2n^2\varepsilon\tau. \tag{5}$$

On the other hand at each step according to (2), there is a decrease of the error accumulated in the previous steps under the exponential law

$$\Delta'_j = \Delta(t-\tau)\, e^{-\pi^2\tau} - \Delta(t-\tau)\,. \tag{6}$$

Taking into account (5) and (6) as an upper bound we obtain the equation

$$\frac{\Delta(t) - \Delta(t-\tau)}{\tau} = 2\varepsilon n^2 - \Delta(t-\tau)\,\frac{1 - e^{-\pi^2\tau}}{\tau}, \qquad \frac{d\Delta(t)}{dt} + \pi^2\Delta(t) = 2\varepsilon n^2,$$

with the solution as the function

$$\Delta(t) = \frac{2\varepsilon n^2}{\pi^2}\left(1 - e^{-\pi^2 t}\right). \tag{7}$$

This dependence can be considered as an upper estimate of error caused by the inaccuracy of the SLAE solution.

There are shown the results of numerical investigation of the real values of the coefficient in front of εn^2 in (7) $C(t) = \Delta(t)/(\varepsilon n^2)$. The calculated results were compared with the accurate solution (2).

The dependences of the coefficient $C(t)$ for $m = 2n$ are given in fig. 1 *a*. The error almost coincides with the estimate.

The curves 1, 2, 3 in fig. 1 *b* corresponds to $\varepsilon = 10^{-10}$, 10^{-8}, 10^{-6}, O is the estimate, the maximum of the real errors is less than estimates approximately by 20%. This is explained as when $\lambda = 1$ the number of iterations is essentially smaller. And the real value $\sigma_j = \left\| x^{(k)} - x^{(k-1)} \right\|$ on the j-th step can be much smaller ε . Therefore, the accumulation of error is slower than it was assumed in the derivation of estimates.

For rather large n and ε the magnitude of the accumulated error $\Delta(t)$ can start to decrease when the number of iterations in step becomes equal to 1. This is explained by the reduction of σ_j from step to step due to the decrease of $u(r,t)$ by exponential low. Fig. 1, b shows the appearance of decay of the error, and with the decrease of ε the point of decay beginning moves to large t (as n grows there is a shift towards smaller t).

Thus, it is found the upper error estimate, which shows its linear dependence on threshold value of limit criteria of iterations number, the quadratic increase of the error from the number of partitions in space and its independence from the number of partitions in time, and is in good agreement with the real errors.

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