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A. L. Rabinovich, **A. L. Talis** (Petrozavodsk, IB KarRC RAS; Moscow, INEOS RAS). **System of channel-like substructures which are mappings from a four-dimensional diamond-like polytope: symmetry.**

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Abstract: A group-theoretical description of the system of channel-like structures has been obtained which are constructed on the basis of mappings from the polytope $\{240\}$ into 3-dimensional Euclidean space using the Hopf fibration. The system of channels was shown to be an "ideal prototype" for real structures.

Keywords: polytope $\{240\}$, channel-like structures, non-crystallographic symmetry.

Eighteen six-fold rings converge in each of the 240 vertices of a diamond-like 4-dimensional polyhedron (polytope $\{240\}$) [1]; this is the densest vertex environment that can be achieved in tetrahedrally coordinated systems. Extreme structural characteristics are attractive for constructing mappings of substructures (especially linear ones) of the polytope $\{240\}$ in 3-dimensional Euclidean space E^3.

It is convenient to obtain mappings using the Hopf fibration [2]: the polytope $\{240\}$ is represented as the product of a "base" and a "fibre", e.g. $\{240\} \rightarrow ([4^6, 6^8])$ (fibre $\{10\}$), where $([4^6, 6^8])$ is the base — Fedorov's parallelohedron (the union of two *right* icosahedrons). Each of the 24 vertices of the base corresponds to a fibre $\{10\}$ consisting of 10 vertices joined by a screw axis 10_1.

The 12 vertices of a *right* icosahedron can be covered by either 4 regular triangles, or 1 regular and 3 isosceles triangles. The triangular face of the icosahedron corresponds to the 30-vertex Berdijk-Coxeter helicoid (tetrahelix) [2]. If all the triangles are regular and each of the four pairs belongs to the same hexagonal face of the base (in this case, centers of the triangles form a regular tetrahedron), then the resulting map in E^3 can be reduced to four "channel-like" structures — they were discussed in [1, 2].

To reveal the maximum possible symmetry of the substructures-mappings, including "channel-like" ones, it is necessary to consider the union of two *regular* icosahedrons (in which all triangles are regular). It was shown that combination of the triangles can be carried out in such a way that one pair of triangles on the base will be on the C_3 axis of order 3, and the remaining 3 pairs of triangles will be converted to each other by this axis. We denote the triangle as "3", and the three triangles defined by $C_3 g_i C_{3v}$ as " 3^3 ", where C_{3v} is the symmetry group of regular triangle, g_i are elements of the regular icosahedron group Y_h which do not belong to C_{3v}. Then, in the *general case*, there is a partition $(3, 3^3; 3, 3^3)$ of 24 vertices of the truncated octahedron, i.e. two 12-vertex subsets, each of which in turn is divided into two subsets $(3, 3^3)$.

A partition $(4 \times 3,\ 4 \times 3)$ of 24 vertices is also possible [3, Tabl.10.3]. The set 4×3 can be considered simultaneously as 4 equivalent, in the symmetry sense, triangles (their centers form a regular tetrahedron - just as in this case, with the symmetry group T_d), and as 3 equivalent "golden" rectangles covering all the icosahedron vertices. The permutation group of all vertices of a rectangle is the symmetrical group S_4, and that of a triangle - S_3, therefore the symmetry group of the set $(4 \times 3,\ 4 \times 3)$ is a direct product $S_4 \times S_3$.

Let us prove that this group is the *maximal* group acting on the set $(4 \times 3,\ 4 \times 3)$. Indeed, the group $S_4 \times S_3$ is the maximal subgroup of the $M_{12} \cdot 2$ group, which order is twice as large as that of the Mathieu group M_{12} [4, p.33] and $M_{12} \cdot 2$ acts (like M_{12}) on a set of 24 points [3, Tabl.10.3]. The group of external automorphisms of $M_{12} \cdot 2$ group is the group $2 \cdot M_{12} \cdot 2$, which already acts on the set of 48 points [3, p.339, 357]; therefore, the group $2 \cdot M_{12} \cdot 2$ does not have a supergroup of the group $S_4 \times S_3$ acting on the set $(4 \times 3,\ 4 \times 3)$ of 24 points, which was to be proved.

The group S_4 is isomorphic to the groups T_d and O (which are subgroups of the octahedron O_h symmetry group), and the group S_3 — to the groups C_{3v} and D_3 (D_3 is the dihedral group). Therefore, in the system under consideration $(4 \times 3,\ 4 \times 3)$, the $T_d \times C_{3v}$ group should be considered the geometric realization of the $S_4 \times S_3$ group. The group T_d is of order 24, C_{3v} is of order 6, so the order of $T_d \times C_{3v}$ is $24 \cdot 6 = 144$. Each vertex of the base with $T_d \times C_{3v}$ symmetry in the polytope' $\{240\}$ Hopf fibration is "loaded" with 10 points of the fiber which are transported to each other by the cyclic group C_{10} of order 10. Therefore, the whole structure corresponds to the $(T_d \times C_{3v}) \cdot C_{10}$ group of order $144 \cdot 10 = 1440$, which is a subgroup of the permutation group S_{240} of the 240 vertices of the polytope $\{240\}$. Since this order is comparable with the polytope' $\{240\}$ order 2880 [1], the combination of 4 channel-like substructures can play the role of an "ideal prototype".

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